Bethe ansatz solution of the $Osp(1|2n)$ invariant spin chain

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Abstract

We have applied the analytical Bethe ansatz approach in order to solve the $Osp(1|2n)$ invariant magnet. By using the Bethe ansatz equations we have calculated the ground state energy and the low-lying dispersion relation. The finite size properties indicate that the model has a central charge $c = n$.

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In the last past years many hierarchies of exactly vertex models have been found [1]. Of particular interest are those invariant by superalgebras $Sl(n|m)$ and $Osp(n|2m)$ in which the associated Boltzmann weights satisfy a graded version of the Yang-Baxter relation [2, 3]. An interesting example of a system in statistical mechanics which can be realized in terms of the $Sl(n|m)$ series is the Perk-Shultz model [4, 5]. Although the Bethe ansatz properties of the $Sl(n|m)$ model have been examined in different contexts [2, 5, 6] in the literature, not much is known concerning the exact solution of the $Osp(n|2m)$ hierarchy. In fact, only recently that the isotropic $Osp(1|2)$ chain [7] (see also ref.[2]) and $U_q Osp(2|2)$ models [8] have been solved in detail by the Bethe ansatz approach. In this sense, we believe that it is important to make an effort in order to find the exact solution of a more general class of $Osp(n|2m)$ invariant magnets.

The purpose of this Letter is to present the analytical Bethe ansatz solution of the $Osp(1|2n)$ spin chain for *any* value of n. This extends previous results found by the author in the simplest case of the $Osp(1|2)$ magnet. The ground state energy, the low-lying dispersion relation and the associated central charge are also computed by using the corresponding Bethe ansatz equations.

The $Osp(1|2n)$ chain is defined by the following Hamiltonian,

$$H = -J \sum_{i=1}^{L} [P_{i,i+1}^{g} + \frac{2}{2n+1} E_{i,i+1}^{g}]$$ (1)

where periodic boundary conditions are assumed, and L is the number of sites of the lattice. Here we are interested in the antiferromagnetic regime $J > 0$ of Hamiltonian (1). The symbols $P_{i,i+1}^{g}$ and $E_{i,i+1}^{g}$ stand for the graded permutation operator [2] and the $Osp(1|2n)$ Temperely-Lieb invariant [7, 10], respectively. More precisely [7, 10], we have the matrix elements $(E_{i,i+1}^{g})_{ab}^{cd} = \alpha_{ab}\alpha_{cd}^{st}$ where the matrix α is defined by

$$\alpha = \begin{pmatrix} 1 & O_{1X2n} \\ O_{2nX1} & \begin{pmatrix} O_{nXn} & I_{nXn} \\ -I_{nXn} & O_{nXn} \end{pmatrix} \end{pmatrix}$$ (2)

where I_{aXa} (O_{aXa}) is the identity (null) aXa matrix.

The problem of diagonalization of Hamiltonian (1) is equivalent to that of finding the Bethe ansatz solution of the corresponding vertex system. For instance, the model (1) is obtained as the logarithmic derivative of the following Transfer matrix $T(\lambda, \eta)$ at $\lambda = 0$,

$$T(\lambda, \eta) = Tr_0[\mathcal{L}_{0L}(\lambda, \eta) \cdots \mathcal{L}_{01}(\lambda, \eta)] \tag{3}$$

where $\mathcal{L}(\lambda, \eta)^{cd}_{ab} = (-1)^{p(a)p(b)} R(\lambda, \eta)^{cd}_{ab}$ [2, 7], and the indices $p(a)$ and 0 denote the Grasman parities [1] and $(2n+1)X(2n+1)$ auxiliary space, respectively. The operator $R(\lambda, \eta)$ is the $Osp(1|2n)$ solution of the graded Yang-Baxter equation given by

$$(R(\lambda, \eta)_{i,i+1})^{cd}_{ab} = \lambda \delta_{ac} \delta_{bd} + \eta (P^g_{i,i+1})^{cd}_{ab} + \frac{\eta \lambda}{(n+1/2)\eta - \lambda} (E^g_{i,i+1})^{cd}_{ab} \tag{4}$$

The first step toward the diagonalization of (3) is to perform a redefinition of the grading to $F \cdots B \cdots F$ by exchanging the first (bosonic) and the $(n+1)$-esimo (fermionic) degrees of freedom. As we shall see below, such canonical transformation has the advantage of adapting the vertex operator $\mathcal{L}(\lambda, \eta)$ in a more symmetric way, before acting it on the reference state. The next step is to notice that the usual ferromagnetic vacuum defined by

$$|0> = \prod_i^L |0>_i; \quad |0>_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5}$$

is an eigenvector of the Transfer-matrix (3). In fact, the vertex $\mathcal{L}(\lambda, \eta)$ has a triangular form when acting on such reference state, namely

$$\mathcal{L}(\lambda, \eta)|0> = \begin{pmatrix} \eta - \lambda & * & * & \cdots & & * \\ 0 & \lambda & * & \cdots & & * \\ 0 & 0 & \lambda & \cdots & & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & \frac{\lambda((n-1/2)\eta - \lambda)}{\lambda - (n+1/2)\eta} \end{pmatrix} \tag{6}$$

[1] In the case of the $BFF \cdots F$ (one boson and 2n fermions) grading we have $p(1) = 0$, $p(i) = 1$ $i = 2, \cdots, 2n + 1$.

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which leads to the following eigenvalue $\Lambda(\lambda)$ of $T(\lambda, \eta)$

$$\Lambda(\lambda) = (\eta - \lambda)^L + \sum_{j=1}^{2n-1} \lambda^L + \left[\frac{\lambda([n-1/2]\eta - \lambda)}{\lambda - (n+1/2)/\eta}\right]^L \tag{7}$$

In accordance to the hypothesis of the analytical Bethe ansatz approach [11], one now seeks for a more general ansatz of form

$$\Lambda(\lambda, \{\lambda_j^k\}) = (\eta - \lambda)^L \prod_{j=1}^{M_1} A(\lambda - \lambda_j^1) + \lambda^L \sum_{k=1}^{2n-1} \prod_{j=1}^{M_k} B(\lambda, \lambda_j^k, \lambda_j^{k+1})$$
$$+ \left[\frac{\lambda([n-1/2]\eta - \lambda)}{\lambda - (n+1/2)\eta}\right]^L \prod_{j=1}^{M_1} C(\lambda - \lambda_j^1) \tag{8}$$

where $A(x)$, $B_j(x)$ and $C(x)$ are some rational functions which can be fixed by using the crossing symmetry, unitarity condition and the asymptotic behaviour of $\Lambda(\lambda)$. In particular, we have found the following relations between these functions

$$C((n+1/2)\eta - x) = A(x), \quad B_{2n-k}((n+1/2)\eta - x) = B_k(x), \quad B_1(x) = A(x - \eta/2)A^{-1}(x - \eta) \tag{9}$$

Taking into account that the amplitude $A(x)$ satisfies the "unitarity" condition, $A(x)A(-x) = 1$, and from our previous experience with the $Osp(1|2)$ case [7], we can set this function as

$$A(x) = -\frac{x + \eta/2}{x - \eta/2} \tag{10}$$

and by using interactively this solution on Eq.(9) we then end up with the following result

$$\Lambda(\lambda, \{\lambda_j^k\}) = (i - \lambda)^L \prod_{j=1}^{M_1} -\frac{\lambda - \lambda_j^1 + i/2}{\lambda - \lambda_j^1 - i/2} + \lambda^L \left\{ \sum_{k=1}^{n-1} \prod_{j=1}^{M_k} -\frac{\lambda - \lambda_j^k - i(k+2)/2}{\lambda - \lambda_j^k - ik/2} \right.$$

$$\prod_{j=1}^{M_{k+1}} -\frac{\lambda - \lambda_j^{k+1} - i(k-1)/2}{\lambda - \lambda_j^{k+1} - i(k+1)/2} + \prod_{j=1}^{M_n} \left(\frac{\lambda - \lambda_j^n - i(n+2)/2}{\lambda - \lambda_j^n - i(n)/2}\right)\left(\frac{\lambda - \lambda_j^n - i(n-1)/2}{\lambda - \lambda_j^n - i(n+1)/2}\right)$$

$$+ \sum_{k=n+1}^{2n-1} \prod_{j=1}^{M_{2n-k}} -\frac{\lambda - \lambda_j^{2n-k} - i(k-1)/2}{\lambda - \lambda_j^{2n-k} - i(k+1)/2} \prod_{j=1}^{M_{2n-k+1}} -\frac{\lambda - \lambda_j^{2n-k+1} - i(k+2)/2}{\lambda - \lambda_j^{2n-k+1} - ik/2} \right\}$$

$$+ \left[\frac{\lambda([n-1/2]i - \lambda)}{\lambda - [n+1/2]i}\right]^L \prod_{j=1}^{M_1} -\frac{\lambda - \lambda_j^1 - i(n+1)}{\lambda - \lambda_j^1 - in} \tag{11}$$

where we conveniently choose $\eta = i$ due to the scale invariance $\lambda \rightarrow \eta\lambda$.

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Finally, the condition that the residues of $\Lambda(\lambda, \{\lambda_j^l\})$ at $\lambda = \lambda_j^l + l/2$, $l = 1, \cdots, n$ vanishes, then fixes the following Bethe ansatz equations for the variables $\{\lambda_j^l\}$ [2]

$$\left(\frac{\lambda_j^1 - i/2}{\lambda_j^1 + i/2}\right)^L = (-1)^{L-M_2} \prod_{k=1}^{M_1} \frac{\lambda_j^1 - \lambda_k^1 - i}{\lambda_j^1 - \lambda_k^1 + i} \prod_{k=1}^{M_2} \frac{\lambda_j^1 - \lambda_k^2 + i/2}{\lambda_j^1 - \lambda_k^2 - i/2}$$

$$\prod_{k=1}^{M_l} \frac{\lambda_j^l - \lambda_k^l - i}{\lambda_j^l - \lambda_k^l + i} = (-1)^{M_{l-1}-M_{l+1}} \prod_{k=1}^{M_{l+1}} \frac{\lambda_j^l - \lambda_k^{l+1} - i/2}{\lambda_j^l - \lambda_k^{l+1} + i/2} \prod_{k=1}^{M_{l-1}} \frac{\lambda_j^l - \lambda_k^{l-1} - i/2}{\lambda_j^l - \lambda_l^{l-1} + i/2}, \quad l = 2, \cdots, n-1$$

$$\prod_{k=1}^{M_n} \left(\frac{\lambda_j^n - \lambda_k^n - i}{\lambda_j^n - \lambda_k^n + i}\right) \left(\frac{\lambda_j^n - \lambda_k^n + i/2}{\lambda_j^n - \lambda_k^n - i/2}\right) = (-1)^{M_{n-1}-M_n} \prod_{k=1}^{M_{n-1}} \frac{\lambda_j^n - \lambda_k^{n-1} - i/2}{\lambda_j^n - \lambda_l^{n-1} + i/2} \tag{12}$$

where the numbers M_l are related to the many sectors indices r_l of the theory by $M_l = L - r_l$. The eigenenergies $E(L)$ of Hamiltonian (1) are parametrized in terms of the Bethe ansatz roots $\{\lambda_j^1\}$ by

$$E(L) = -\sum_{j=1}^{M_1} \frac{1}{(\lambda_j^1)^2 + 1/4} + L \tag{13}$$

An interesting characteristic of these equations is the appearance of the phase factors ± 1, distinguishing the behaviour of roots $\{\lambda_j^l\}$ in the sectors r_l of the system. Here we mention that similar factors have also been found in the solution of the Perk-Shultz model [5]. In particular, for the simplest case of the $Osp(1|2)$ chain, such phase plays the role of an index which physical meaning is to split the even and odd degrees of freedom present in the theory [7]. In any case, such phase factors will change the logarithm branches of the Bethe ansatz equations (12) and therefore they are extremely important in the correct characterization of the ground state and the low-lying excitations.

Let us turn to the computation of some properties in the thermodynamic limit of Hamiltonian (1). We have found that the ground state in a given sector r_l is parametrized by a set of real roots $\{\lambda_j^l\}$ of the Bethe ansatz equations. In this case, by taking the logarithm of Eqs.(12) we find that

$$L\phi_{1/2}(\lambda_j^1) = 2\pi Q_j^1 + \sum_{k=1,k\neq j}^{M_1} \phi_1(\lambda_j^1 - \lambda_k^1) - \sum_{k=1}^{M_2} \phi_{1/2}(\lambda_j^1 - \lambda_k^2)$$

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[2] Analogously, the crossing symmetry (last two terms of Eq.(11)) guarantee that the extra poles at $\lambda = \lambda_j^{2n+1-l} + l/2$, $l = n+1, \cdots, 2n$ produces the same restriction (12) for the set $\{\lambda_j^l\}$.

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$$\sum_{k=1,k\neq j}^{M_l} \phi_1(\lambda_j^l - \lambda_k^l) + 2\pi Q_j^l = \sum_{k=1}^{M_{l-1}} \phi_{1/2}(\lambda_j^l - \lambda_k^{l-1}) + \sum_{k=1}^{M_{l+1}} \phi_{1/2}(\lambda_j^l - \lambda_k^{l+1}), \ l = 2, \cdots, n-1$$

$$\sum_{k=1,k\neq j}^{M_n} [\phi_1(\lambda_j^n - \lambda_k^n) - \phi_{1/2}(\lambda_j^n - \lambda_k^n)] + 2\pi Q_j^n = \sum_{k=1}^{M_{n-1}} \phi_{1/2}(\lambda_j^n - \lambda_k^{n-1}) \quad (14)$$

where $\phi_a(x) = 2\arctan(x/a)$ and Q_j^l are the following numbers characterizing the different branches of the logarithm

$$Q_j^l = -\frac{[L - r_l - 1]}{2} + j - 1, \quad j = 1, 2, \cdots, L - r_l \quad (15)$$

In the thermodynamic limit, $L \to \infty$, the roots $\{\lambda_j^l\}$ cluster into a continuous distribution of densities $\rho_l(\lambda)$ satisfying a set of n-coupled integral equations. This system of integral equations can be solved by elementary Fourier techniques, and here we only summarize our results

$$\rho_l(\lambda) = \frac{4}{2n+1} \frac{\cos[\frac{(2n+1-2l)\pi}{2(2n+1)}]\cosh[\frac{2\lambda\pi}{(2n+1)}]}{\cosh[\frac{4\lambda\pi}{2n+1}] + \cos[\frac{(2n+1-2l)\pi}{2n+1}]} \quad (16)$$

The ground state per particle is calculated by using density $\rho_1(\lambda)$ in Eq.(13), after replacing the sum by an integral. The final result is

$$e_\infty = -\int_{-\infty}^{\infty} \frac{\rho_1(\lambda)}{\lambda^2 + 1/4} d\lambda + 1 = 1 - \frac{2}{2n+1} \{2\ln(2) + \psi[1/2 + 1/(2n+1)] - \psi[1/(2n+1)]\}$$

$$(17)$$

where $\psi(x)$ is the Euler psi-function. The low-lying excitation over the ground state are obtained by the insertion of "holes " on the distribution of the numbers Q_j^l. In general, this generates n branches of excitations and we find that all of them are gapless. More precisely, the low-momentum p dispersion relation has the behaviour

$$\epsilon_l(p) = \frac{2\pi}{2n+1} p, l = 1, \cdots, n \quad (18)$$

and therefore we find a unique sound velocity $v_s = \frac{2\pi}{2n+1}$.

To conclude we would like to present some numerical results of the finite size behaviour of the ground state of Hamiltonian (1). Such finite size effects can be explicitly related

to the central charge governing the underlying conformal field theory of this system. For instance, by extrapolating the following sequence [12],

$$\frac{E(L)}{L} = e_\infty - \frac{\pi v_s c}{6L^2} \tag{19}$$

we are able to compute the central charge c. In order to do that, we have solved numerically the Bethe ansatz equations up $L = 44$ and by substituting the solution $\{\lambda_j^1\}$ in Eq.(13) we have determined the ground state energy for finite L. In Table 1, we present our results for the estimatives (19) in the case of $n = 2$ and $n = 3$. Taking into account these results and those from our previous study of the $Osp(1|2)$ chain ($c = 1$), we conjecture that the underlying central charge is $c = n$. In spite of our numerical results, this conjecture is also confirmed by using the analytical method of refs. [13, 14], since we have n-nested Bethe ansatz equations of *real* roots. One possible physical interpretation of this result is as follows. Recalling that this system is made by one bosonic and $2n$ fermionic degrees of freedom, we may conclude that only the fermions contribute to the central charge, namely $c = (2n)/2 = n$.

In summary we have presented the Bethe ansatz solution, the ground state and the low-lying dispersion relation of the $Osp(1|2n)$ spin chain. Interesting enough, we have noticed that the ground state structure resembles much that appearing in the $O(2n + 3)(B_{n+1})$ [15] invariant magnets. We believe that this fact indicates that (as we have shown for the $Osp(1|2)$ model [7]) the $Osp(1|2n)$ chain can be obtained as a peculiar and *new* branch limit of the anisotropic A_{2n-1}^2 vertex model. Our results together with those of ref. [16] for the conformal anomaly c, strongly suggest that the anisotropic A_{2n-1}^2 model has in fact many gapless regimes. We hope to address such questions in future publications.

Acknowledgements

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Table Captions

Table 1. The estimative of the conformal anomaly from equation (19).

Table 1

L	$n = 2$	n=3
8	1.97 108	2.85 851
16	1.99 703	2.97 544
24	2.00 032	2.99 170
32	2.00 118	2.99 675
40	2.00 144	2.99 887
48	2.00 152	2.99 991
Extrapolated	2.00 0(3)	3.00 0(1)